

TOPOLOGICAL PROPERTIES OF Ad-SEMISIMPLE CONJUGACY CLASSES IN LIE GROUPS

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ABSTRACT. We prove that Ad-semisimple conjugacy classes in a connected Lie group G are closed embedded submanifolds of G . We also prove that if $\alpha : H \rightarrow G$ is a homomorphism of connected Lie groups such that the kernel of α is discrete in H , then for an Ad-semisimple conjugacy class C in G , every connected component of $\alpha^{-1}(C)$ is a conjugacy class in H . Corresponding results for adjoint orbits in real Lie algebras are also proved.

1. INTRODUCTION

Conjugacy classes in linear algebraic groups are extensively studied and have had many applications (see, for example, [6]). We recall two well-known results concerning topological properties of semisimple conjugacy classes in algebraic groups below (see [5, Proposition 18.2] as well as the proofs of [6, Theorem 3.8] and [8, Lemma 1.10]).

Fact 1.1. *Let \mathbf{G} be a linear algebraic group defined over an algebraically closed field. Then semisimple conjugacy classes in \mathbf{G} are Zariski closed.*

Fact 1.2. *Let \mathbf{G} be a linear algebraic group defined over an algebraically closed field, \mathbf{H} a Zariski closed subgroup of \mathbf{G} . Then for every semisimple conjugacy class C in \mathbf{G} , $C \cap \mathbf{H}$ is a finite union of conjugacy classes in \mathbf{H} .*

As for Lie groups which are isomorphic to the groups of \mathbb{R} -points of certain algebraic groups defined over \mathbb{R} , we have the following result (see [3, Proposition 10.1]).

Fact 1.3. *Let $\mathbf{G} \subset GL_n(\mathbb{C})$ be a linear algebraic group defined over \mathbb{R} , $G = \mathbf{G} \cap GL_n(\mathbb{R})$. Then semisimple conjugacy classes in G are closed with respect to the Hausdorff topology.*

But for general Lie groups, there are relatively less known results concerning properties of conjugacy classes, partly because of the lack of algebraic instruments and the nonlinearity involved in the problems.

The purpose of this paper is to prove the Lie-theoretic counterparts of Facts 1.1 and 1.2. The corresponding results for adjoint orbits in real Lie algebras are also obtained. Since the notion of semisimplicity makes no sense for elements in general Lie groups or general real Lie algebras, we introduce the following notions.

Let G be a Lie group with Lie algebra \mathfrak{g} . An element g of G is *Ad-semisimple* if $\text{Ad}(g)$ is semisimple in $GL(\mathfrak{g})$. An element X of \mathfrak{g} is *ad-semisimple* if $\text{ad}(X)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$. A conjugacy class C in G is *Ad-semisimple* if a (hence every)

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element of C is Ad-semisimple. An adjoint orbit O in \mathfrak{g} is ad-semisimple if a (hence every) element of O is ad-semisimple.

The main results in this paper are the following two assertions.

Theorem 1.1. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . Then*

- (i) *Ad-semisimple conjugacy classes in G are closed embedded submanifolds of G ;*
- (ii) *ad-semisimple adjoint orbits in \mathfrak{g} are closed embedded submanifolds of \mathfrak{g} .*

Theorem 1.2. *Let $\alpha : H \rightarrow G$ be a homomorphism of connected Lie groups. Suppose $\ker(\alpha)$ is discrete in H . Let C be an Ad-semisimple conjugacy class in G , and let O be an ad-semisimple adjoint orbit in the Lie algebra \mathfrak{g} of G . Then*

- (i) *Every connected component of $\alpha^{-1}(C)$ is a conjugacy class in H ;*
- (ii) *Every connected component of $(d\alpha)^{-1}(O)$ is an adjoint orbit in the Lie algebra \mathfrak{h} of H .*

The following corollary of Theorem 1.2 is obvious.

Corollary 1.3. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , and let H be a connected closed subgroup of G with Lie algebra \mathfrak{h} . Then*

- (i) *For every Ad-semisimple conjugacy class C in G , every connected component of $C \cap H$ is a conjugacy class in H ;*
- (ii) *For every ad-semisimple adjoint orbit O in \mathfrak{g} , every connected component of $O \cap \mathfrak{h}$ is an adjoint orbit in \mathfrak{h} .*

Theorem 1.1 generalizes Fact 1.3. The closedness of conjugacy classes of finite order in connected Lie groups, which is a special case of item (i) of Theorem 1.1, was proved in [1] using real analytic geometry.

The proofs of Facts 1.1 and 1.2 are based on computations of dimensions of subvarieties of G and \mathfrak{g} . But such computations do not help to prove Theorems 1.1 and 1.2 because of the facts: (1) the dimension of the boundary of an immersed submanifold of a manifold is not necessarily strictly smaller than the dimension of the submanifold; (2) the intersection of two submanifolds of a manifold is not necessarily a submanifold.

Basically, our proofs of Theorems 1.1 and 1.2 are Lie-theoretic. But the utilization of some facts of real algebraic groups is essential, including the multiplicative Jordan decomposition on real algebraic groups and Whitney's Theorem on the finiteness of the number of connected components of a real algebraic variety (see the proofs of Lemmas 3.6, 3.7, and 3.8 below).

The starting point of our proofs of Theorems 1.1 and 1.2 is the following assertion.

Theorem 1.4. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , $\rho : G \rightarrow GL(\mathcal{V})$ a representation of G in a finite dimensional real vector space \mathcal{V} . Suppose $\ker(\rho)$ is discrete in G . Let $f \in \mathbb{R}[\lambda]$ be a real polynomial without multiple roots in \mathbb{C} . Then*

- (i) *Every connected component of the subset*

$$Z(\rho, f) = \{g \in G \mid f(\rho(g)) = 0\}$$

of G is a closed embedded submanifold of G , and is a conjugacy class in G ;

- (ii) *Every connected component of the subset*

$$\mathfrak{z}(d\rho, f) = \{X \in \mathfrak{g} \mid f(d\rho(X)) = 0\}$$

of \mathfrak{g} is a closed embedded submanifold of \mathfrak{g} , and is an adjoint orbit in \mathfrak{g} .

For a conjugacy class C in a connected Lie group G , and an adjoint orbit O in the Lie algebra \mathfrak{g} of G , define the sets

$$\Gamma(C) = g^{-1}C \cap Z(G), \quad g \in C,$$

$$\gamma(O) = (-X + O) \cap Z(\mathfrak{g}), \quad X \in O,$$

where $Z(G)$ and $Z(\mathfrak{g})$ are the centers of G and \mathfrak{g} , respectively. It can be shown that $\Gamma(C)$ and $\gamma(O)$ are independent of the choices of $g \in C$ and $X \in O$, and that $\Gamma(C)$ is a subgroup of $Z(G)$, $\gamma(O)$ is an additive subgroup of $Z(\mathfrak{g})$.

The proofs of Theorems 1.1 and 1.2 are also based on the following result.

Theorem 1.5. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . We have*

- (i) *If C is an Ad-semisimple conjugacy class in G , then $\Gamma(C)$ is a finite subgroup of $Z(G)$;*
- (ii) *If O is an ad-semisimple adjoint orbit in \mathfrak{g} , then $\gamma(O)$ is trivial.*

Now we give a sketch of the contents of the following sections. In Section 2 we prove Theorem 1.4. The key point is the construction of a map $\varphi_g : \mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \rightarrow G$ for $g \in Z(\rho, f)$, which maps some neighborhood U of $0 \in \mathfrak{g}$ onto a neighborhood of g diffeomorphically, where $\mathfrak{a}_1 = \ker(1 - \text{Ad}(g))$, $\mathfrak{a}_2 = \text{Im}(1 - \text{Ad}(g))$. Then one can show that for $Y \in U$, $\varphi_g(Y) \in Z(\rho, f)$ if and only if $Y \in \mathfrak{a}_2$ if and only if $\varphi_g(Y)$ is conjugate to g . This idea is essentially due to N.-H. Neeb. The proof in the Lie algebra level is similar.

After showing some basic properties of $\Gamma(C)$ and $\gamma(O)$, Theorem 1.5 is proved in Section 3. We first prove two special cases of item (i) of the theorem for which G is linear or semisimple using properties of real algebraic groups, and then reduce the general case of item (i) to these two cases using Levi's decomposition. The finiteness of $\Gamma(C)$ is mainly deduced from Whitney's Theorem which states that the number of connected components of a real algebraic variety is finite. The proof of item (ii) makes use of item (i). One can also give a direct proof of item (ii), which is similar to that of item (i).

Theorem 1.1 is proved in Section 4. We first use Theorem 1.4 to prove that the projection of an Ad-semisimple conjugacy class C in G on $G/Z(G)_0$ is closed. This implies that $C \cdot Z(G)_0$ is closed. By Theorem 1.5, one can show that $C \cdot Z(G)_0$ is a fiber bundle over $Z(G)_0/(\Gamma(C) \cap Z(G)_0)$, and the fiber above the identity is just C . So C is closed. The proof of the closedness of ad-semisimple adjoint orbits is similar.

In Section 5 we prove Theorem 1.2. If G is linear and α is injective, item (i) of Theorem 1.2 can be easily deduced from Theorem 1.4. We first prove the theorem under the assumptions that $\Gamma(C)$ is trivial and α is injective using this observation by considering the adjoint group of G , and then deduce the general case from this. The proof of item (ii) is similar but much easier.

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2. CHARACTERIZATIONS OF CONJUGACY CLASSES BY POLYNOMIALS

In this section we prove the following theorem, which is the starting point of the proofs of Theorems 1.1 and 1.2.

Theorem 2.1. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , $\rho : G \rightarrow GL(\mathcal{V})$ a representation of G in a finite dimensional real vector space \mathcal{V} . Suppose $\ker(\rho)$ is*

discrete. Let $f \in \mathbb{R}[\lambda]$ be a real polynomial without multiple roots in \mathbb{C} . Then

(i) Every connected component of the subset

$$Z(\rho, f) = \{g \in G \mid f(\rho(g)) = 0\}$$

of G is a closed embedded submanifold of G , and is a conjugacy class in G ;

(ii) Every connected component of the subset

$$\mathfrak{z}(d\rho, f) = \{X \in \mathfrak{g} \mid f(d\rho(X)) = 0\}$$

of \mathfrak{g} is a closed embedded submanifold of \mathfrak{g} , and is an adjoint orbit in \mathfrak{g} .

Proof. (i) Denote $Z = Z(\rho, f)$. Firstly, we note that Z is invariant under the conjugation of G . So Z is the union of some conjugacy classes in G .

Let $g \in Z$. By the definition of the set Z , $f(\rho(g)) = 0$. Since f has no multiple roots, $\rho(g)$ is semisimple. We claim that g is Ad-semisimple. Indeed, since $\ker(\rho)$ is discrete, the differential $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathcal{V})$ of ρ is injective. So the action of $\text{Ad}(g)$ on \mathfrak{g} is equivalent to the action of $\text{Ad}(\rho(g))|_{d\rho(\mathfrak{g})}$ on $d\rho(\mathfrak{g})$. Since $\rho(g)$ acts semisimply on \mathcal{V} , $\text{Ad}(\rho(g))$ acts semisimply on $\mathfrak{gl}(\mathcal{V})$, and then $\text{Ad}(\rho(g))|_{d\rho(\mathfrak{g})}$ acts semisimply on $d\rho(\mathfrak{g})$. This verifies the claim.

Denote $\mathfrak{a}_1 = \ker(1 - \text{Ad}(g))$, $\mathfrak{a}_2 = \text{Im}(1 - \text{Ad}(g))$. Since $\text{Ad}(g)$ is semisimple, $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Define a map $\varphi_g : \mathfrak{a}_1 \oplus \mathfrak{a}_2 \rightarrow G$ by

$$\varphi_g(Y_1, Y_2) = e^{Y_2} e^{Y_1} g e^{-Y_2}, \quad Y_1 \in \mathfrak{a}_1, Y_2 \in \mathfrak{a}_2.$$

Then it is easy to compute the differential $(d\varphi_g)_{(0,0)} : \mathfrak{a}_1 \oplus \mathfrak{a}_2 \rightarrow T_g G$ of φ_g at $(0, 0)$ as

$$(d\varphi_g)_{(0,0)}(Y_1, Y_2) = (dr_g)_e(Y_1 + (1 - \text{Ad}(g))Y_2),$$

where r_g is the right translation on G induced by g . Since $\text{Ad}(g)$ is semisimple, the restriction of $1 - \text{Ad}(g)$ on $\mathfrak{a}_2 = \text{Im}(1 - \text{Ad}(g))$ is a linear automorphism. Hence $(d\varphi_g)_{(0,0)}$ is a linear isomorphism. By the Implicit Function Theorem, there exist an open neighborhood U_1 of $0 \in \mathfrak{a}_1$ and an open neighborhood U_2 of $0 \in \mathfrak{a}_2$ such that the restriction of φ_g to $U_1 \times U_2 \subset \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is a diffeomorphism onto an open neighborhood $U = \varphi_g(U_1 \times U_2)$ of $g \in G$.

Define a map $\alpha_g : \mathfrak{a}_1 \rightarrow \mathfrak{gl}(\mathcal{V})$ by

$$\alpha_g(Y_1) = f(\rho(e^{Y_1}g)).$$

We claim that α_g is an immersion at $0 \in \mathfrak{a}_1$. Indeed, we have

$$\begin{aligned} & (d\alpha_g)_0(Y_1) \\ &= \left. \frac{d}{dt} \right|_{t=0} \alpha_g(tY_1) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(e^{td\rho(Y_1)}\rho(g)) \\ &= d\rho(Y_1)\rho(g)f'(\rho(g)), \end{aligned}$$

where f' is the derivative of f . Here the last step holds because $e^{td\rho(Y_1)}$ commutes with $\rho(g)$. Since f has no multiple roots, $(f, f') = 1$. So there exist polynomials r, s such that $fr + f's = 1$. Substitute $\rho(g)$ for the indeterminate in this equality and notice that $f(\rho(g)) = 0$, we get $f'(\rho(g))s(\rho(g)) = 1$. So $f'(\rho(g))$ is invertible. Since $\rho(g)$ is also invertible and $d\rho$ is injective, $(d\alpha_g)_0(Y_1) = 0$ implies $Y_1 = 0$. Hence α_g is an immersion at $0 \in \mathfrak{a}_1$. Thus, shrinking U_1 if necessary, we may assume that $\alpha_g|_{U_1}$ is injective.

Now for $Y_1 \in U_1$, $Y_2 \in U_2$, we have

$$\begin{aligned} & f(\rho(\varphi_g(Y_1, Y_2))) \\ &= f(\rho(e^{Y_2})\rho(e^{Y_1}g)\rho(e^{Y_2})^{-1}) \\ &= \rho(e^{Y_2})f(\rho(e^{Y_1}g))\rho(e^{Y_2})^{-1} \\ &= \rho(e^{Y_2})\alpha_g(Y_1)\rho(e^{Y_2})^{-1}. \end{aligned}$$

So $\varphi_g(Y_1, Y_2) \in Z \Leftrightarrow Y_1 = 0$, that is,

$$Z \cap U = \varphi_g(\{0\} \times U_2) = \{e^{Y_2}ge^{-Y_2} | Y_2 \in U_2\}.$$

This shows that every connected component of Z is an embedded submanifold of G , which is necessarily closed by the definition of Z , and that every conjugacy class contained in Z is open in Z . But the connectedness of G implies that conjugacy classes are connected. Hence every conjugacy class contained in Z is in fact a connected component of Z . This proves (i).

(ii) Similar to the proof of (i), the set $\mathfrak{z} = \mathfrak{z}(d\rho, f)$ is the union of some adjoint orbits in \mathfrak{g} . Let $X \in \mathfrak{z}$. Then $d\rho(X)$ and $\text{ad}(X)$ are semisimple. Denote $\mathfrak{b}_1 = \ker(\text{ad}(X))$, $\mathfrak{b}_2 = \text{Im}(\text{ad}(X))$. Then $\mathfrak{g} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Define a map $\psi_X : \mathfrak{b}_1 \oplus \mathfrak{b}_2 \rightarrow \mathfrak{g}$ by

$$\psi_X(W_1, W_2) = \text{Ad}(e^{W_2})(X + W_1), \quad W_1 \in \mathfrak{b}_1, W_2 \in \mathfrak{b}_2.$$

Then

$$(d\psi_X)_{(0,0)}(W_1, W_2) = W_1 - \text{ad}(X)(W_2).$$

Hence $(d\psi_X)_{(0,0)}$ is a linear isomorphism, and then there exist an open neighborhood V_1 of $0 \in \mathfrak{b}_1$ and an open neighborhood V_2 of $0 \in \mathfrak{b}_2$ such that the restriction of ψ_X to $V_1 \times V_2 \subset \mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a diffeomorphism onto an open neighborhood $V = \psi_X(V_1 \times V_2)$ of $X \in \mathfrak{g}$.

Define $\beta_X : \mathfrak{b}_1 \rightarrow \mathfrak{gl}(\mathcal{V})$ by

$$\beta_X(W_1) = f(d\rho(X + W_1)).$$

Then

$$\begin{aligned} & (d\beta_X)_0(W_1) \\ &= \left. \frac{d}{dt} \right|_{t=0} \beta_X(tW_1) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(d\rho(X) + td\rho(W_1)) \\ &= d\rho(W_1)f'(d\rho(X)). \end{aligned}$$

Similar to the proof of (i), we can prove $f'(d\rho(X))$ is invertible. So $(d\beta_X)_0(W_1) = 0$ implies $W_1 = 0$, that is, β_X is an immersion at $0 \in \mathfrak{b}_1$. Shrinking V_1 if necessary, we may assume that $\beta_X|_{V_1}$ is injective.

Now for $W_1 \in V_1$, $W_2 \in V_2$, we have

$$\begin{aligned} & f(d\rho(\psi_X(W_1, W_2))) \\ &= f(d\rho(\text{Ad}(e^{W_2})(X + W_1))) \\ &= f(\rho(e^{W_2})d\rho(X + W_1)\rho(e^{W_2})^{-1}) \\ &= \rho(e^{W_2})f(d\rho(X + W_1))\rho(e^{W_2})^{-1} \\ &= \rho(e^{W_2})\beta_X(W_1)\rho(e^{W_2})^{-1}. \end{aligned}$$

So $\psi_X(W_1, W_2) \in \mathfrak{z} \Leftrightarrow W_1 = 0$, that is,

$$\mathfrak{z} \cap V = \psi_X(\{0\} \times V_2) = \{\text{Ad}(e^{W_2})(X) | W_2 \in V_2\}.$$

Then an argument similar to the proof of (i) shows that every connected component of \mathfrak{z} is a closed embedded submanifold of \mathfrak{g} , and is an adjoint orbit. This proves (ii). \square

Corollary 2.2. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , and let $\rho : G \rightarrow GL(\mathcal{V})$ be a representation of G in a finite dimensional real vector space \mathcal{V} . Suppose $\ker(\rho)$ is discrete. We have*

(i) *If C is a conjugacy class in G such that $\rho(C)$ contains a semisimple element A of $GL(\mathcal{V})$, then C is a closed embedded submanifold of G , and is a connected component of the set*

$$Z = \{g \in G | f(\rho(g)) = 0\},$$

where f is the minimal polynomial of A ;

(ii) *If O is an adjoint orbit in \mathfrak{g} such that $d\rho(O)$ contains a semisimple element B of $\mathfrak{gl}(\mathcal{V})$, then O is a closed embedded submanifold of \mathfrak{g} , and is a connected component of the set*

$$\mathfrak{z} = \{X \in \mathfrak{g} | p(d\rho(X)) = 0\},$$

where p is the minimal polynomial of B . \square

3. FINITENESS OF $\Gamma(C)$

Let G be a connected Lie group with Lie algebra \mathfrak{g} . Let C be a conjugacy class in G , and let O be an adjoint orbit in \mathfrak{g} . The subset $\Gamma(C) = \Gamma_G(C)$ of the center $Z(G)$ of G is defined by

$$\Gamma_G(C) = g^{-1}C \cap Z(G), \quad g \in C.$$

The subset $\gamma(O) = \gamma_{\mathfrak{g}}(O)$ of the center $Z(\mathfrak{g})$ of \mathfrak{g} is defined by

$$\gamma_{\mathfrak{g}}(O) = (-X + O) \cap Z(\mathfrak{g}), \quad X \in O.$$

For convenience, denote

$$\Gamma_0(C) = \Gamma(C) \cap Z(G)_0 = g^{-1}C \cap Z(G)_0, \quad g \in C,$$

where $Z(G)_0$ is the identity component of $Z(G)$. In this section we prove some properties of $\Gamma(C)$ and $\gamma(O)$, especially the finiteness of $\Gamma(C)$ and the triviality of $\gamma(O)$ under the Ad-semisimplicity or ad-semisimplicity condition.

Lemma 3.1. *Let G be a connected Lie group, C a conjugacy class in G , O an adjoint orbit in the Lie algebra \mathfrak{g} of G . Then*

- (i) *$\Gamma(C)$ is independent of the choice of the element $g \in C$ defining it;*
- (ii) *$\gamma(O)$ is independent of the choice of the element $X \in O$ defining it.*

Proof. (i) Let $g_1, g_2 \in C$. Then $g_1 = hg_2h^{-1}$ for some $h \in G$. Hence we have

$$\begin{aligned} g_1^{-1}C \cap Z(G) &= hg_2^{-1}h^{-1}C \cap Z(G) = hg_2^{-1}(h^{-1}Ch)h^{-1} \cap Z(G) \\ &= hg_2^{-1}Ch^{-1} \cap Z(G) = h(g_2^{-1}C \cap Z(G))h^{-1} = g_2^{-1}C \cap Z(G). \end{aligned}$$

This proves (i).

(ii) Let $X_1, X_2 \in O$. Then $X_1 = \text{Ad}(g)X_2$ for some $g \in G$. Hence

$$\begin{aligned} (-X_1 + O) \cap Z(\mathfrak{g}) &= (-\text{Ad}(g)X_2 + O) \cap Z(\mathfrak{g}) \\ &= \text{Ad}(g)(-X_2 + O) \cap Z(\mathfrak{g}) = \text{Ad}(g)((-X_2 + O) \cap Z(\mathfrak{g})) \\ &= (-X_2 + O) \cap Z(\mathfrak{g}). \end{aligned}$$

This proves (ii). \square

For an element g in a connected Lie group G , we denote by $Z_G(g)$ the centralizer of g in G , and denote

$$N_G(g) = \{h \in G \mid g^{-1}hgh^{-1} \in Z(G)\}.$$

$N_G(g)$ is a closed subgroup of G containing $Z_G(g)$. In fact, if we let $\pi : G \rightarrow G/Z(G)$ be the quotient homomorphism, then $N_G(g) = \pi^{-1}(Z_{G/Z(G)}(\pi(g)))$. Similarly, for an element X in the Lie algebra \mathfrak{g} of G , denote by $Z_G(X)$ the centralizer of X in G , and denote

$$N_G(X) = \{h \in G \mid -X + \text{Ad}(h)X \in Z(\mathfrak{g})\}.$$

Then $N_G(X) = \pi^{-1}(Z_{G/Z(G)}(d\pi(X)))$ is a closed subgroup of G containing $Z_G(X)$.

Lemma 3.2. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , C a conjugacy class in G , O an adjoint orbit in \mathfrak{g} . Then*

- (i) $\Gamma(C)$ is a Lie subgroup of $Z(G)$, and is isomorphic to $N_G(g)/Z_G(g)$ for every $g \in C$;
- (ii) $\gamma(O)$ is a Lie subgroup of the vector group $Z(\mathfrak{g})$, and is isomorphic to $N_G(X)/Z_G(X)$ for every $X \in O$.

Proof. (i) Let $g \in C$. Define a smooth map $\alpha : N_G(g) \rightarrow Z(G)$ by

$$\alpha(h) = g^{-1}hgh^{-1}.$$

We claim that α is a homomorphism of Lie groups. Indeed, let $h_1, h_2 \in N_G(g)$, then

$$\begin{aligned} \alpha(h_1)\alpha(h_2) &= (g^{-1}h_1gh_1^{-1})(g^{-1}h_2gh_2^{-1}) \\ &= g^{-1}h_1g(g^{-1}h_2gh_2^{-1})h_1^{-1} = g^{-1}(h_1h_2)g(h_1h_2)^{-1} \\ &= \alpha(h_1h_2). \end{aligned}$$

It is obvious that the kernel of α is $Z_G(g)$, and the image of α is $\Gamma(C) = g^{-1}C \cap Z(G)$. So $\Gamma(C)$ is a Lie subgroup of $Z(G)$, and is isomorphic to $N_G(g)/Z_G(g)$.

(ii) Let $X \in O$. Define $\beta : N_G(X) \rightarrow Z(\mathfrak{g})$ by

$$\beta(h) = -X + \text{Ad}(h)X.$$

For $g_1, g_2 \in N_G(X)$, we have

$$\begin{aligned} \beta(g_1g_2) &= -X + \text{Ad}(g_1g_2)X \\ &= (-X + \text{Ad}(g_1)X) + (-\text{Ad}(g_1)X + \text{Ad}(g_1)\text{Ad}(g_2)X) \\ &= \beta(g_1) + \text{Ad}(g_1)(-X + \text{Ad}(g_2)X) \\ &= \beta(g_1) + \text{Ad}(g_1)\beta(g_2) = \beta(g_1) + \beta(g_2). \end{aligned}$$

So β is a homomorphism of Lie groups. The kernel of β is $Z_G(X)$, the image of β is $\gamma(O) = (-X + O) \cap Z(\mathfrak{g})$. So $\gamma(O)$ is a Lie subgroup of $Z(\mathfrak{g})$, and is isomorphic to $N_G(X)/Z_G(X)$. \square

For a connected Lie group G with Lie algebra \mathfrak{g} and an adjoint orbit O in \mathfrak{g} , $\exp(O)$ is a conjugacy class in G . $\gamma(O)$ and $\Gamma(\exp(O))$ have the following relation.

Lemma 3.3. *Let G be a connected Lie group with Lie algebra \mathfrak{g} , O an adjoint orbit in \mathfrak{g} . Then $\exp(\gamma(O)) \subset \Gamma_0(\exp(O))$.*

Proof. Let $X \in O$. If $Y \in \gamma(O)$, then there exists $h \in G$ such that $Y = -X + \text{Ad}(h)X$. Since $Y \in Z(\mathfrak{g})$, $he^X h^{-1} = e^{\text{Ad}(h)X} = e^{X+Y} = e^X e^Y$. So $e^Y = e^{-X} h e^X h^{-1} \in e^{-X} \exp(O) \cap Z(G)_0 = \Gamma_0(\exp(O))$. This shows $\exp(\gamma(O)) \subset \Gamma_0(\exp(O))$. \square

Let $\pi : G \rightarrow G'$ be a covering homomorphism of Lie groups. Then for a conjugacy class C in G , $\pi(C)$ is a conjugacy class in G' . The next lemma relates $\Gamma_G(C)$ with $\Gamma_{G'}(\pi(C))$.

Lemma 3.4. *Let $\pi : G \rightarrow G'$ be a covering homomorphism of connected Lie groups, C a conjugacy class in G . Then $\pi(\Gamma_G(C)) = \Gamma_{G'}(\pi(C))$.*

Proof. First we claim that $Z(G) = \pi^{-1}(Z(G'))$. Indeed, let $z \in \pi^{-1}(Z(G'))$, and let $\alpha : G \rightarrow G$ be the map defined by $\alpha(h) = hzh^{-1}z^{-1}$. Then $\alpha(G) \subset \ker(\pi)$. Since $\alpha(G)$ is connected containing the identity e of G , and $\ker(\pi)$ is discrete, we have $\alpha(G) = \{e\}$. So $z \in Z(G)$. This shows $\pi^{-1}(Z(G')) \subset Z(G)$. It is obvious that $Z(G) \subset \pi^{-1}(Z(G'))$. Hence $Z(G) = \pi^{-1}(Z(G'))$. Now we choose a $g \in C$, then

$$\begin{aligned} \pi(\Gamma_G(C)) &= \pi(g^{-1}C \cap Z(G)) = \pi(g^{-1}C \cap \pi^{-1}(Z(G'))) \\ &= \pi(g^{-1}C) \cap Z(G') = \pi(g)^{-1}\pi(C) \cap Z(G') = \Gamma_{G'}(\pi(C)). \end{aligned}$$

\square

The following lemma demonstrates a rough understanding of $\Gamma(C)$ and $\gamma(O)$ under the semisimplicity assumptions.

Lemma 3.5. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . We have*

- (i) *If C is an Ad-semisimple conjugacy class in G , $g \in C$, then the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide, and $\Gamma(C)$ is a 0-dimensional Lie subgroup of $Z(G)$;*
- (ii) *If O is an ad-semisimple adjoint orbit in \mathfrak{g} , $X \in O$, then the Lie algebras of $N_G(X)$ and $Z_G(X)$ coincide, and $\gamma(O)$ is a 0-dimensional Lie subgroup of the vector group $Z(\mathfrak{g})$.*

Proof. (i) Since $Z_G(g) \subset N_G(g)$, to prove their Lie algebras coincide, it is sufficient to show that for every X in the Lie algebra of $N_G(g)$, X belongs to the Lie algebra of $Z_G(g)$. For such an X , we have $g^{-1}e^{tX}ge^{-tX} \in Z(G)$ for every $t \in \mathbb{R}$. So $e^{tX}e^{-t\text{Ad}(g)X} = g(g^{-1}e^{tX}ge^{-tX})g^{-1} \in Z(G)$. This implies that $(1 - \text{Ad}(g))X$ belongs to the Lie algebra of $Z(G)$, and then $(1 - \text{Ad}(g))^2X = 0$. Since C is Ad-semisimple, $\text{Ad}(g)$ is semisimple. So we in fact have $(1 - \text{Ad}(g))X = 0$. But the Lie algebra of $Z_G(g)$ is $\ker(1 - \text{Ad}(g))$. So X belongs to the Lie algebra of $Z_G(g)$. Hence the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide. As the image of the homomorphism α constructed in the proof of Lemma 3.2, $\Gamma(C)$ is a 0-dimensional Lie subgroup of $Z(G)$.

(ii) Similar to the proof of (i), let Y be an element of the Lie algebra of $N_G(X)$. Then $-X + \text{Ad}(e^{tY})X \in Z(\mathfrak{g})$ for every $t \in \mathbb{R}$. This implies that $\text{ad}(Y)X \in Z(\mathfrak{g})$. So $\text{ad}(X)^2Y = -\text{ad}(X)(\text{ad}(Y)X) = 0$. Since X is ad-semisimple, $\text{ad}(X)Y = 0$. This shows that Y belongs to the Lie algebra of $Z_G(X)$. So the Lie algebras of $N_G(X)$ and $Z_G(X)$ coincide, and $\gamma(O)$ is a 0-dimensional Lie subgroup of $Z(\mathfrak{g})$. \square

Remark 3.1. We only need the discreteness of $\Gamma(C)$ in $Z(G)$ in the proofs of items (i) of Theorems 1.1 and 1.2. By Lemma 3.5, $\Gamma(C)$ is 0-dimensional when C is Ad-semisimple. But this does not imply that $\Gamma(C)$ is discrete in $Z(G)$. To get the discreteness of $\Gamma(C)$, we have to show that it is finite. In fact, if $\Gamma(C)$ could be infinite for some connected Lie group G and some Ad-semisimple conjugacy class C in G , we would easily construct a discrete central subgroup D of $G \times \mathbb{R}$ such that $\Gamma_{(G \times \mathbb{R})/D}(\pi(C))$ is not discrete, where $\pi : G \rightarrow (G \times \mathbb{R})/D$ is the covering homomorphism.

The remaining of this section is devoted to the proof of Theorem 1.5. Some results on real algebraic groups are needed. For convenience, we understand the Zariski topology on $GL_n(\mathbb{R})$ as the topology for which a closed set is the set of common zeros of a family of real polynomial functions on $GL_n(\mathbb{R})$ with indeterminates g_{ij} ($1 \leq i, j \leq n$) and $\frac{1}{\det g}$, where $g = (g_{ij}) \in GL_n(\mathbb{R})$. It is obvious that if G is a Lie subgroup of $GL_n(\mathbb{R})$, then the Zariski closure \overline{G} of G is also a Lie subgroup of $GL_n(\mathbb{R})$.

Lemma 3.6. *Let G be a connected Lie subgroup of $GL_n(\mathbb{R})$ for some n , \overline{G} the Zariski closure of G in $GL_n(\mathbb{R})$. If $g \in G$ is Ad-semisimple in G , then it is Ad-semisimple in \overline{G} .*

Proof. Let $g = g_s g_u$ be the multiplicative Jordan decomposition of g in $GL_n(\mathbb{R})$, where g_s is semisimple, g_u is unipotent. It is well known that $g_s, g_u \in \overline{G}$ (see, for example, [2, Chapter 1, Section 4]). Then $\text{Ad}_{\overline{G}}(g) = \text{Ad}_{\overline{G}}(g_s) \cdot \text{Ad}_{\overline{G}}(g_u)$ is the multiplicative Jordan decomposition of $\text{Ad}_{\overline{G}}(g)$ in $GL(\overline{\mathfrak{g}})$, where $\overline{\mathfrak{g}}$ is the Lie algebra of \overline{G} . Since the Lie algebra \mathfrak{g} of G is invariant under $\text{Ad}_{\overline{G}}(g)$, it is also invariant under $\text{Ad}_{\overline{G}}(g_s)$ and $\text{Ad}_{\overline{G}}(g_u)$. So $\text{Ad}(g) = \text{Ad}_{\overline{G}}(g_s)|_{\mathfrak{g}} \cdot \text{Ad}_{\overline{G}}(g_u)|_{\mathfrak{g}}$ is the multiplicative Jordan decomposition of $\text{Ad}(g)$ in $GL(\mathfrak{g})$. But by the assumption, $\text{Ad}(g)$ is semisimple. So $\text{Ad}_{\overline{G}}(g_u)|_{\mathfrak{g}} = 0$. This implies that $G \subset Z_{\overline{G}}(g_u)$. Since $Z_{\overline{G}}(g_u)$ is Zariski closed, we have $\overline{G} \subset Z_{\overline{G}}(g_u)$, that is, $g_u \in Z(\overline{G})$. So $\text{Ad}_{\overline{G}}(g_u) = 1$, and then $\text{Ad}_{\overline{G}}(g) = \text{Ad}_{\overline{G}}(g_s)$ is semisimple, that is, g is Ad-semisimple in \overline{G} . \square

Lemma 3.7. *Let G be a connected Lie subgroup of $GL_n(\mathbb{R})$ for some n , C an Ad-semisimple conjugacy class in G . Then $\Gamma(C)$ is a finite subgroup of $Z(G)$.*

Proof. Let \overline{G} be the Zariski closure of G in $GL_n(\mathbb{R})$, and let C' be the conjugacy class in \overline{G} containing C . Choose a $g \in C$. Since g is Ad-semisimple in G , by Lemma 3.6, g is Ad-semisimple in \overline{G} . By Lemma 3.5, the Lie algebras of $N_{\overline{G}}(g)$ and $Z_{\overline{G}}(g)$ coincide. Since $N_{\overline{G}}(g)$ can be expressed as

$$N_{\overline{G}}(g) = \{h \in \overline{G} \mid (g^{-1} h g h^{-1})x = x(g^{-1} h g h^{-1}), \forall x \in \overline{\mathfrak{g}}\},$$

which is algebraic, by Whitney's Theorem [10], $N_{\overline{G}}(g)$ has finitely many connected components. So as a quotient group of the component group of $N_{\overline{G}}(g)$, $N_{\overline{G}}(g)/Z_{\overline{G}}(g)$ is finite. Hence $\Gamma_{\overline{G}}(C') \cong N_{\overline{G}}(g)/Z_{\overline{G}}(g)$ is finite.

We claim that $Z(G) \subset Z(\overline{G})$. Indeed, if $z \in Z(G)$, then $Z_{GL_n(\mathbb{R})}(z)$ is an algebraic subgroup of $GL_n(\mathbb{R})$ containing G . So $Z_{GL_n(\mathbb{R})}(z)$ contains \overline{G} , that is,

$z \in Z(\overline{\overline{G}})$. This shows $Z(G) \subset Z(\overline{\overline{G}})$. Now we have

$$\Gamma_G(C) = g^{-1}C \cap Z(G) \subset g^{-1}C' \cap Z(\overline{\overline{G}}) = \Gamma_{\overline{\overline{G}}}(C').$$

Hence $\Gamma_G(C)$ is finite. This proves the lemma. \square

Lemma 3.8. *Let G be a connected semisimple Lie group, C an Ad-semisimple conjugacy class in G . Then $\Gamma(C)$ is a finite subgroup of $Z(G)$.*

Proof. Let $\text{Aut}(\mathfrak{g})$ be the automorphism group of the Lie algebra \mathfrak{g} of G . Since

$$\text{Aut}(\mathfrak{g}) = \{A \in GL(\mathfrak{g}) \mid f_{X,Y}(A) = 0, \forall X, Y \in \mathfrak{g}\},$$

where

$$f_{X,Y}(A) = A[X, Y] - [AX, AY]$$

is algebraic, $\text{Aut}(\mathfrak{g})$ is an algebraic subgroup of $GL(\mathfrak{g})$. Choose $g \in C$. Then $\text{Ad}(g) \in \text{Aut}(\mathfrak{g})$. By Whitney's Theorem, $Z_{\text{Aut}(\mathfrak{g})}(\text{Ad}(g))$ has finitely many connected components. Since G is semisimple, $\text{Ad}(G)$ is the identity component of $\text{Aut}(\mathfrak{g})$. So $Z_{\text{Ad}(G)}(\text{Ad}(g)) = Z_{\text{Aut}(\mathfrak{g})}(\text{Ad}(g)) \cap \text{Ad}(G)$ has finitely many connected components. Since the kernel of the epimorphism $\text{Ad} : G \rightarrow \text{Ad}(G)$ is $Z(G)$, which is discrete, $N_G(g)/Z(G) \cong Z_{\text{Ad}(G)}(\text{Ad}(g))$ has finitely many connected components.

On the other hand, we have

$$\Gamma(C) \cong N_G(g)/Z_G(g) \cong (N_G(g)/Z(G))/(Z_G(g)/Z(G)).$$

By Lemma 3.5, the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide. So the Lie algebras of $N_G(g)/Z(G)$ and $Z_G(g)/Z(G)$ coincide. We have shown that $N_G(g)/Z(G)$ has finitely many connected components. So $\Gamma(C) \cong (N_G(g)/Z(G))/(Z_G(g)/Z(G))$ is finite. \square

Now we can prove the finiteness of $\Gamma(C)$ and the triviality of $\gamma(O)$ under the semisimplicity assumptions.

Theorem 3.9. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . We have*

- (i) *If C is an Ad-semisimple conjugacy class in G , then $\Gamma(C)$ is a finite subgroup of $Z(G)$;*
- (ii) *If O is an ad-semisimple adjoint orbit in \mathfrak{g} , then $\gamma(O)$ is trivial.*

Proof. (i) By Lemma 3.4, we may assume that G is simply connected. Let $R = \text{Rad}(G)$. By Levi's Theorem, there is a connected semisimple subgroup L of G such that $G = R \rtimes L$. Note that R and L are simply connected.

We first prove that $R \cap \Gamma(C)$ is finite. Let $\Lambda(L)$ be the linearizer of L (by definition, $\Lambda(L)$ is the intersection of the kernels of all finite dimensional representations of L). By considering the adjoint representation of L in the Lie algebra of G , we know that $\Lambda(L) \subset Z(G)$. Since $L/\Lambda(L)$ admits a finite dimensional faithful representation (see [7, Chapter 5, Section 3, Theorem 8]), by a theorem of Harish-Chandra [4], $G/\Lambda(L) \cong R \rtimes (L/\Lambda(L))$ admits a finite dimensional faithful representation. Since C is Ad-semisimple in G , $\pi(C)$ is Ad-semisimple in $G/\Lambda(L)$, where $\pi : G \rightarrow G/\Lambda(L)$ is the quotient homomorphism. By Lemma 3.7, $\Gamma_{G/\Lambda(L)}(\pi(C))$ is finite. Since $\Lambda(L)$ is discrete, by Lemma 3.4, $\pi(\Gamma(C)) = \Gamma_{G/\Lambda(L)}(\pi(C))$ is finite. Since $R \cap \Lambda(L)$ is trivial, the restriction of π to $R \cap \Gamma(C)$ is injective. So $R \cap \Gamma(C)$ is finite.

Now consider the quotient homomorphism $\alpha : G \rightarrow G/R$. Since $\alpha(C)$ is Ad-semisimple in G/R , by Lemma 3.8, $\Gamma_{G/R}(\alpha(C))$ is finite. But the kernel of the

homomorphism $\alpha|_{\Gamma(C)} : \Gamma(C) \rightarrow \Gamma_{G/R}(\alpha(C))$ is $R \cap \Gamma(C)$, which we have shown is finite. So $\alpha(C)$ is finite. This proves (i).

(ii) We may assume that G is simply connected. Since O is ad-semisimple, $\exp(O)$ is an Ad-semisimple conjugacy class in G . By item (i) of the theorem, $\Gamma(\exp(O))$ is finite. So $\Gamma_0(\exp(O)) = \Gamma(\exp(O)) \cap Z(G)_0$ is a finite subgroup of $Z(G)_0$. But the simple connectedness of G implies that $Z(G)_0$ is simply connected (see [9, Corollary 3.18.6]), which is isomorphic to a vector group. So $\Gamma_0(\exp(O))$ is in fact trivial. By Lemma 3.3, $\exp(\gamma(O))$ is trivial. But the simple connectedness of $Z(G)_0$ implies that the restriction of the exponential map to $Z(\mathfrak{g})$ is injective. In particular, $\exp|_{\gamma(O)}$ is injective. So $\gamma(O)$ is trivial. \square

Remark 3.2. Our proof of item (ii) of Theorem 3.9 is based on item (i) of that theorem. But one can also give a direct proof of item (ii). To do this, one can embed \mathfrak{g} into some $\mathfrak{gl}_n(\mathbb{R})$ using Ado's Theorem, consider the connected Lie subgroup G' of $GL_n(\mathbb{R})$ with Lie algebra \mathfrak{g} , and then consider the Zariski closure $\overline{G'}$ of G' . In this course one need a result similar to Lemma 3.6, that is, if $X \in \mathfrak{g}$ is ad-semisimple in \mathfrak{g} , then it is ad-semisimple in the Lie algebra of $\overline{G'}$. The details are similar to the proof of Lemma 3.7 and are omitted here.

4. PROOF OF THEOREM 1.1

Based on the preparations of the previous two sections, in this section we prove the closedness of Ad-semisimple conjugacy classes in connected Lie groups and ad-semisimple adjoint orbits in real Lie algebras.

Theorem 4.1. *Let G be a connected Lie group with Lie algebra \mathfrak{g} . Then*

- (i) *Ad-semisimple conjugacy classes in G are closed embedded submanifold of G ;*
- (ii) *ad-semisimple adjoint orbits in \mathfrak{g} are closed embedded submanifold of \mathfrak{g} .*

Proof. Let $G' = G/Z(G)_0$, where $Z(G)_0$ is the identity component of the center $Z(G)$ of G . Let $\pi : G \rightarrow G'$ be the quotient homomorphism. Then the adjoint representation $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ induces naturally a representation $\rho : G' \rightarrow GL(\mathfrak{g})$, such that $\rho \circ \pi = \text{Ad}$. Note that $\ker(\rho)$ is discrete in G' .

(i) Let C be an Ad-semisimple conjugacy class in G . Then $C' = \pi(C)$ is a conjugacy class in G' . Since all elements of $\rho(C') = \text{Ad}(C)$ are semisimple in $GL(\mathfrak{g})$, by Corollary 2.2, C' is a closed embedded submanifold of G' . So

$$M = \pi^{-1}(C') = C \cdot Z(G)_0$$

is a closed embedded submanifold of G .

Now consider the transitive action of $G \times Z(G)_0$ on the manifold M , defined by

$$(h, z).x = h x h^{-1} z.$$

Choose a $g \in C \subset M$, and let $L \subset G \times Z(G)_0$ be the isotropic group of g . Then the map

$$\varphi : (G \times Z(G)_0)/L \rightarrow M$$

defined by

$$\varphi((h, z)L) = h g h^{-1} z$$

is a diffeomorphism.

By Theorem 3.9, $\Gamma(C) = g^{-1}C \cap Z(G)$ is a finite subgroup of $Z(G)$. So $\Gamma_0(C) = g^{-1}C \cap Z(G)_0 = \Gamma(C) \cap Z(G)_0$ is a finite subgroup of $Z(G)_0$. Then $Z(G)_0/\Gamma_0(C)$ is a Lie group. Let

$$\alpha : G \times Z(G)_0 \rightarrow Z(G)_0/\Gamma_0(C)$$

be the epimorphism defined by

$$\alpha(h, z) = [z],$$

where $[z]$ is the image of z under the quotient homomorphism $Z(G)_0 \rightarrow Z(G)_0/\Gamma_0(C)$. For $(h, z) \in L$, $hgh^{-1}z = (h, z).g = g$, so $z^{-1} = g^{-1}hgh^{-1} \in \Gamma_0(C)$, and then $[z]$ is trivial in $Z(G)_0/\Gamma_0(C)$. This shows that $L \subset \ker(\alpha)$. Then α induces a smooth map

$$\tilde{\alpha} : (G \times Z(G)_0)/L \rightarrow Z(G)_0/\Gamma_0(C)$$

defined by

$$\tilde{\alpha}((h, z)L) = [z].$$

It is obvious that $(G \times Z(G)_0)/L$ is a fiber bundle with base space $Z(G)_0/\Gamma_0(C)$, fiber type $\tilde{\alpha}^{-1}([e])$, and projection $\tilde{\alpha}$.

We claim that

$$\tilde{\alpha}^{-1}([e]) = \varphi^{-1}(C).$$

Firstly, let $(h, z)L \in \tilde{\alpha}^{-1}([e])$. Then $[z] = [e]$, that is, $z \in \Gamma_0(C)$. So there exists $k \in G$ such that $z = g^{-1}kgk^{-1}$. Then

$$\varphi((h, z)L) = hgh^{-1}z = hgz h^{-1} = hg(g^{-1}kgk^{-1})h^{-1} = (hk)g(hk)^{-1} \in C,$$

that is, $(h, z)L \in \varphi^{-1}(C)$. Conversely, let $(h', z')L \in \varphi^{-1}(C)$. Then there exists $k' \in G$ such that $\varphi((h', z')L) = h'gz'h'^{-1} = k'gk'^{-1}$. This implies $z' = g^{-1}(h'^{-1}k')g(h'^{-1}k')^{-1}$. So $z' \in \Gamma_0(C)$. Hence $(h', z')L \in \tilde{\alpha}^{-1}([e])$. This verifies the claim.

As the fiber above $[e]$, $\varphi^{-1}(C) = \tilde{\alpha}^{-1}([e])$ is a closed embedded submanifold of $(G \times Z(G)_0)/L$. Since φ is a diffeomorphism, C is a closed embedded submanifold of M , hence a closed embedded submanifold of G . Item (i) is proved.

(ii) Let O be an ad-semisimple adjoint orbit in \mathfrak{g} . Then $O' = d\pi(O)$ is an adjoint orbit in \mathfrak{g}' , the Lie algebra of G' . Since all elements of $d\rho(O') = \text{ad}(O)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$, by Corollary 2.2, O' is a closed embedded submanifold of \mathfrak{g}' . So

$$N = (d\pi)^{-1}(O') = O + Z(\mathfrak{g})$$

is a closed embedded submanifold of \mathfrak{g} .

Consider the transitive action of $G \times Z(\mathfrak{g})$ on the manifold N , defined by

$$(h, Y).W = \text{Ad}(h)W + Y.$$

Choose an $X \in O \subset N$, and let $K \subset G \times Z(\mathfrak{g})$ be the isotropic group of X . Then the map

$$\psi : (G \times Z(\mathfrak{g}))/K \rightarrow N$$

defined by

$$\psi((h, Y)K) = \text{Ad}(h)X + Y$$

is a diffeomorphism.

Let

$$\beta : G \times Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g})$$

be the projection to the second factor. For $(h, Y) \in K$, $\text{Ad}(h)X + Y = (h, Y).X = X$, so $-Y = -X + \text{Ad}(h)X \in \gamma(O)$. But by Theorem 3.9, $\gamma(O)$ is trivial. So $Y = 0$. This shows that $K \subset \ker(\beta)$. Then β induces a smooth map

$$\tilde{\beta} : (G \times Z(\mathfrak{g}))/K \rightarrow Z(\mathfrak{g})$$

defined by

$$\tilde{\beta}((h, Y)K) = Y.$$

It is obvious that $(G \times Z(\mathfrak{g}))/K$ is a fiber bundle with base space $Z(\mathfrak{g})$, fiber type $\tilde{\beta}^{-1}(0)$, and projection $\tilde{\beta}$. Similar to the proof of (i), we have $\tilde{\beta}^{-1}(0) = \psi^{-1}(O)$.

As the fiber above $0 \in Z(\mathfrak{g})$, $\psi^{-1}(O) = \tilde{\beta}^{-1}(0)$ is a closed embedded submanifold of $(G \times Z(\mathfrak{g}))/K$. Since ψ is a diffeomorphism, O is a closed embedded submanifold of N , hence a closed embedded submanifold of \mathfrak{g} . This proves (ii). \square

5. PROOF OF THEOREM 1.2

We give the proof of Theorem 1.2 in this section. We first prove a lemma.

Lemma 5.1. *Let $\pi : G \rightarrow G'$ be a covering homomorphism of connected Lie groups. If C' is an Ad-semisimple conjugacy class in G' , then every connected component of $\pi^{-1}(C')$ is a conjugacy class in G .*

Proof. By Theorem 4.1, C' is a closed embedded submanifold of G' . Let \tilde{C} be a connected component of $\pi^{-1}(C')$. Then \tilde{C} is a closed embedded submanifold of G . Since G is connected, \tilde{C} is invariant under the conjugation of G . Let C be a conjugacy class in G contained in \tilde{C} . By Theorem 4.1, C is a closed embedded submanifold of G , hence a closed embedded submanifold of \tilde{C} . But $\dim C = \dim C' = \dim \tilde{C}$. By the connectedness of \tilde{C} , we must have $C = \tilde{C}$. \square

Remark 5.1. Lemma 5.1 does not hold without the Ad-semisimplicity assumption.

Theorem 5.2. *Let $\alpha : H \rightarrow G$ be a homomorphism of connected Lie groups. Suppose $\ker(\alpha)$ is discrete. Let the Lie algebras of G and H be \mathfrak{g} and \mathfrak{h} , respectively. We have*

- (i) *If C is an Ad-semisimple conjugacy class in G , then every connected component of $\alpha^{-1}(C)$ is a conjugacy class in H ;*
- (ii) *If O is an ad-semisimple adjoint orbit in \mathfrak{g} , then every connected component of $(d\alpha)^{-1}(O)$ is an adjoint orbit in \mathfrak{h} .*

Proof. (i) We first observed that $\alpha^{-1}(C)$ is invariant under the conjugation of H . So $\alpha^{-1}(C)$ is the union of a family of conjugacy classes in H . But the connectedness of H implies that conjugacy classes in H are connected. So every connected component of $\alpha^{-1}(C)$ is the union of a family of conjugacy classes in H . We prove that every connected component of $\alpha^{-1}(C)$ is a conjugacy class in H . The proof is divided into three steps.

Step (a). We prove (i) under the additional assumptions that α is injective and $\Gamma(C)$ is trivial. In this case, H can be identified with $\alpha(H)$, which is a Lie subgroup of G , and then $\alpha^{-1}(C)$ is identified with $C \cap H$. Note that under this identification, the prior topology on H may be different from the subspace topology on H induced from G . We call the prior topology on H the *H-topology*, and call a connected component of $C \cap H$ with respect to the *H-topology* an *H-connected component* of $C \cap H$.

Let C_i be an H -connected component of $C \cap H$. Consider the adjoint homomorphism $\text{Ad}_G = \text{Ad} : G \rightarrow \text{Ad}(G)$. Then $\text{Ad}_G(C_i) \subset \text{Ad}_G(C) \cap \text{Ad}_G(H)$. Let C'_i be the $\text{Ad}_G(H)$ -connected component of $\text{Ad}_G(C) \cap \text{Ad}_G(H)$ containing $\text{Ad}_G(C_i)$. Since all elements of $\text{Ad}_G(C)$ are semisimple, by Corollary 2.2, the conjugacy class $\text{Ad}_G(C)$ in $\text{Ad}(G)$ is an $\text{Ad}(G)$ -connected component of $Z = \{A \in \text{Ad}(G) | f(A) = 0\}$, where f is the minimal polynomial of $\text{Ad}_G(h_0)$ for some $h_0 \in H$. So C'_i is an $\text{Ad}_G(H)$ -connected component of $Z \cap \text{Ad}_G(H)$. By Corollary 2.2 again, we conclude that C'_i is a conjugacy class in $\text{Ad}_G(H)$. Let $g_1, g_2 \in C_i$. Then $\text{Ad}_G(g_1), \text{Ad}_G(g_2) \in C'_i$, and then there exists $h \in H$ such that $\text{Ad}_G(g_2) = \text{Ad}_G(h)\text{Ad}_G(g_1)\text{Ad}_G(h)^{-1}$. So $g_2 = hg_1h^{-1}z$ for some $z \in Z(G)$. But g_1 and g_2 are conjugate in G . So there is $g \in G$ such that $g_2 = gg_1g^{-1}$. This implies $gg_1g^{-1} = hg_1h^{-1}z = hg_1zh^{-1}$. Hence $z = g_1^{-1}(h^{-1}g)g_1(h^{-1}g)^{-1} \in g_1^{-1}C \cap Z(G) = \Gamma(C)$. But we have assumed that $\Gamma(C)$ is trivial. So $z = e$, and then $g_2 = hg_1h^{-1}$. This shows that C_i is a conjugacy class in H .

Step (b). We prove (i) under the additional assumption that α is injective. As we have done in step (a), we identify H with $\alpha(H)$. Let $G' = G/\Gamma(C)$. By Theorem 3.9, $\Gamma(C)$ is finite. So the quotient homomorphism $\pi : G \rightarrow G'$ is a covering homomorphism. In particular, $\pi|_H : H \rightarrow \alpha(H)$ is a covering homomorphism. Let C_i be an H -connected component of $C \cap H$, and let C' be a conjugacy class in H contained in C_i . Then $\pi(C')$ is a conjugacy class in $\pi(H)$, and we have $\pi(C') \subset \pi(C_i) \subset \pi(C) \cap \pi(H)$. Let C'_i be the $\pi(H)$ -connected component of $\pi(C) \cap \pi(H)$ containing $\pi(C_i)$. By Lemma 3.4, $\Gamma_{G'}(\pi(C)) = \pi(\Gamma(C))$ is trivial. So by step (a), C'_i is a conjugacy class in $\pi(H)$ containing $\pi(C')$. This forces $\pi(C') = \pi(C_i) = C'_i$. Hence $C' \subset C_i \subset (\pi|_H)^{-1}(C'_i)$. By Lemma 5.1, C' is an H -connected component of $(\pi|_H)^{-1}(C'_i)$. As an H -connected subset of $(\pi|_H)^{-1}(C'_i)$ containing C' , C_i must coincide with C' . So C_i is a conjugacy class in H .

Step (c). We finish the proof of item (i). Let C_i be a connected component of $\alpha^{-1}(C)$, and let C'_i be the $\alpha(H)$ -connected component of $C \cap \alpha(H)$ containing $\alpha(C_i)$. Then C_i is a connected component of $\alpha^{-1}(C'_i)$. But by step (b), C'_i is a conjugacy class in $\alpha(H)$. So by Lemma 5.1, C_i is a conjugacy class in H .

(ii) Since $d\alpha$ is injective, \mathfrak{h} can be viewed as a subalgebra of \mathfrak{g} . We want to prove that if $O \cap \mathfrak{h}$ is nonempty, then every connected component of $O \cap \mathfrak{h}$ is an adjoint orbit in \mathfrak{h} . Let O_i be a connected component of $O \cap \mathfrak{h}$. Then $\text{ad}_{\mathfrak{g}}(O_i) \subset \text{ad}_{\mathfrak{g}}(O) \cap \text{ad}_{\mathfrak{g}}(\mathfrak{h})$. Let O'_i be the connected component of $\text{ad}_{\mathfrak{g}}(O) \cap \text{ad}_{\mathfrak{g}}(\mathfrak{h})$ containing $\text{ad}_{\mathfrak{g}}(O_i)$. Since all elements of $\text{ad}_{\mathfrak{g}}(O)$ are semisimple, by Corollary 2.2, the adjoint orbit $\text{ad}_{\mathfrak{g}}(O)$ in $\text{ad}(\mathfrak{g})$ is a connected component of $\mathfrak{z} = \{B \in \text{ad}(\mathfrak{g}) | p(B) = 0\}$, where p is the minimal polynomial of $\text{ad}_{\mathfrak{g}}(Y_0)$ for some $Y_0 \in \mathfrak{h}$. So O'_i is a connected component of $\mathfrak{z} \cap \text{ad}_{\mathfrak{g}}(\mathfrak{h})$. By Corollary 2.2 again, we conclude that O'_i is an adjoint orbit in $\text{ad}_{\mathfrak{g}}(\mathfrak{h})$. Let $X_1, X_2 \in O_i$. Then $\text{ad}_{\mathfrak{g}}(X_1), \text{ad}_{\mathfrak{g}}(X_2) \in O'_i$, and then there exists $h \in H$ such that $\text{ad}_{\mathfrak{g}}(X_2) = \text{Ad}(\text{Ad}_G(h))\text{ad}_{\mathfrak{g}}(X_1)$. So $X_2 = \text{Ad}_G(h)X_1 + Y$ for some $Y \in Z(\mathfrak{g})$. But X_1 and X_2 lie in the same adjoint orbit in \mathfrak{g} . So there is $g \in G$ such that $X_2 = \text{Ad}_G(g)X_1$. This implies $Y = -\text{Ad}_G(h)X_1 + \text{Ad}_G(g)X_1 \in \gamma(O)$. By Theorem 3.9, $\gamma(O)$ is trivial. So $Y = 0$, and then $X_2 = \text{Ad}_G(h)X_1 + Y$. This shows that O_i is an adjoint orbit in \mathfrak{h} . (ii) is proved. \square

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